Stresses at Junction of Cone and Cylinder in Tanks with Cone Bottoms or Ends

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EDITORS NOTE: The war has stimulated the building of large, heavy pressure vessels of all kinds and shapes. A number of these have cone heads, or cone bottoms, of a size considerably larger than encountered a few years ago. Steel plate designers have long known that compression stresses exist at the junction of a cylindrical shell with a cone bottom and have made calculations of such stresses. The manner of providing reinforcing and the extent of reinforcing to be used, however, has always been a question. The ASME and API- ASME Codes are silent, both as to the stresses themselves and the manner and extent of reinforcing to be provided. To the best of our knowledge, a satisfactory solution has never been worked out. The subject is necessarily rather involved and designers have been satisfied with the use of approximate methods. In the following article, Mr. Boardman has endeavored to give the problem the consideration that it deserves.

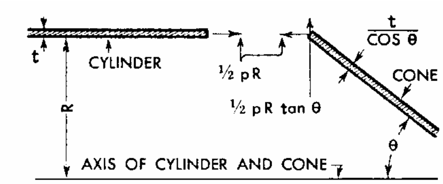

Figure I

p = Internal gas pressure in lbs. per sq. in.
R = Radius of cylinder in inches.
t = Thickness of cylinder wall in inches.

Fig. I represents the end of a strip of cylinder 1 in. wide circumferentially adjoining the end of a strip of cone 1 in. wide circumferentially. Under an internal gas pressure of " p" lbs. per sq. in., cylinder and cone strip ends of the relative thicknesses shown would obviously move radially outward (perpendicular to the cylinder and cone axis) the same distance provided a radially *outward* force of (1/2pR tan θ) acted as indicated. In fact, however, no such force exists. Therefore the cone and cylinder together must resist a radially *inward* force of (1/2pR tan θ) per inch of circumference of the cone-cylinder junction circle. To determine the resulting stresses, which are eventually to be combined with the membrane stresses, it is necessary to set up and solve equations stating mathematically that the end of the cylinder strip and the end of the cone strip must deflect and rotate together. Let:

F = Radial load on end of cone strip in lbs.
f= Radial load on end of cylinder strip in lbs.
M =Moment on adjoining ends of cone and cylinder strips in lb.—ins.

Fig. II shows F, f, and M acting on the strip ends.

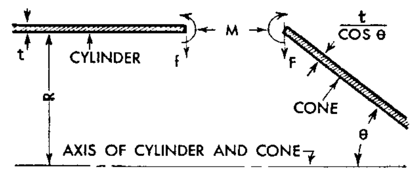

Figure II

The theory of beams on elastic foundations (See Chapter I, Part II, Timoshenko's STRENGTH OF MATERIALS) leads to the *practically accurate* expressions for the ends of the cylinder and cone strips shown in Table I below.

Using these values and Figures I and II the following equations may be written:

(I) $F+f = 1/2pR \tan\theta = Z$

(II) $Fz\cos^2\theta - Mz\beta\cos^2\theta = fz - Mz\beta$ or $F\cos^2\theta - M\beta\cos^2\theta = f - M\beta$

Radial deflection (perpendicular to axis) of end of cone strip due to " F" minus radial deflection of end of cone strip due to " M" = radial deflection of end of cylinder strip due to " f" minus radial deflection of end of cylinder strip due to "M."

(III) $Fz\cos^2\theta - 2Mz\beta^2\cos^2\theta = 2Mz\beta^2 - fz\beta$ or $F\cos^2\theta = f - M\beta$

Rotation of end of cone strip due to "F" minus rotation of end of cone strip due to "M"=rotation of end of cylinder strip due to "M" minus rotation of end of cylinder strip due to " f."

The solutions of equations (I), (II) and (111) are:

(A) $f = ZV_1$
(B) $F = Z(1 - V_1)$
(C) $M = Z\left(\frac{2}{\beta}\right)V_2$

in which

$$Z = \tfrac{1}{2}\,pR\tan\theta \qquad V_1 = \frac{\cos^2\theta(3+\cos^2\theta)}{1+\cos^2\theta(6+\cos^2\theta)}$$

$$\beta = \frac{1.316}{\sqrt{Rt}} \qquad V_2 = \frac{V_1}{3+\cos^2\theta}$$

Table I

	1 Lb. Load on End Perpendicular to Axis		Unit Moment on End	
	Cylinder Strip	Cone Strip	Cylinder Strip	Cone Strip
Radial deflection perpendicular to Axis (ins.)	z	$z\cos^2\Theta$	$z\beta$	$z\beta\cos^2\Theta$
Rotation (radians)	$z\beta$	$z\beta\cos^2\Theta$	$2z\beta^2$	$2z\beta^2\cos^2\Theta$
Moment (lb.-ins.)	0.0	0.0	1.0	1.0
Shear (lbs.)	1.0	$\cos\Theta$	0.0	0.0
Circumferential Stress (lbs. per sq. in.)	$z\left(\frac{m}{R}\right)$	$z\left(\frac{m}{R}\right)\cos^2\Theta$	$z\beta\left(\frac{m}{R}\right)$	$z\beta\left(\frac{m}{R}\right)\cos^2\Theta$

m = Modulus of Elasticity; $\beta = \frac{1.316^*}{\sqrt{Rt}}$; $z = \frac{1}{2mI\beta^3}$; $I = \frac{t^3}{12}$.

*This factor might vary from 1.285 to 1.316 depending upon choice of designer as to use and value of Poisson's Ratio.

Table II

θ	V_1	$1-V_1$	V_2
5°	.499	.501	.125
10°	.496	.504	.125
15°	.491	.509	.125
20°	.484	.516	.125
25°	.475	.525	.124
30°	.464	.536	.124
35°	.450	.550	.123
40°	.433	.567	.121
45°	.412	.588	.118
50°	.386	.614	.113
55°	.355	.645	.106
60°	.317	.683	.098

The longitudinal bending stress at the end of the cylinder strip = $M \div \frac{t^2}{6} = \frac{6M}{t^2}$. The membrane longitudinal stress is $\left(\frac{pR}{2t}\right)$. Therefore the combined bending and direct stresses at the end of the cylinder strip=

$$\frac{pR}{2t} + \frac{6M}{t^2} = \frac{pR}{t}\left(.5 + 4.559\,V_2\sqrt{\frac{R}{t}}\tan\theta\right)$$

The circumferential stress (See Table I) at end of cylinder strip due to "f" and

$$\text{“M”} = fz\left(\frac{m}{R}\right) - Mz\beta\left(\frac{m}{R}\right) = \frac{pR}{t}\left[1.316\sqrt{\frac{R}{t}}\left(V_1 - 2V_2\right)\tan\theta\right]$$

The membrane circumferential tension at the end of the cylinder $= \frac{pR}{t}$. Therefore the total circumferential stress at the end of the cylinder $= \frac{pR}{t}\left[1.316\sqrt{\frac{R}{t}}\left(V_1 - 2V_2\right)\tan\theta\right]$, a minus sign indicating compression.

Let $X = 4.559(\tan\theta)V_2$ and $Y = 1.316\tan\theta(V_1 - 2V_2)$. Then a general expression for the longitudinal stress at the end of the cylinder is $\frac{pR}{t}\left(.5 + X\frac{R}{t}\right)$; and a general expression for the circumferential stress at the end of the cylinder is $\frac{pR}{t}\left(1 - Y\frac{R}{t}\right)$

Table III summarizes the foregoing analysis, and also gives values of (.0120°) and (.0050°) which are practical equivalents of X and Y, respectively.

It is apparent from a study of the expression $\frac{pR}{t}\left(.5 + \frac{R}{t}\right)$ that the longitudinal stresses are very high for pressure vessels of common proportions. However, these stresses are of the same nature as those in the hubs of bolted flanged connections, and do not exist, in the intensities calculated, after the application of the test pressure. For this reason it seems reasonable to ignore them in designing, and to think of the plastic longitudinal deformation under test as the final fabricating operation.

It also seems reasonable to permit at the cone-cylinder junction a circumferential compression stress of $\frac{pR}{t}$ as *calculated* from the expression $\frac{pR}{t}\left(1 - Y\frac{R}{t}\right)$, even though it is realized that the value thus calculated is in error because of the above discussed plastic yielding in the longitudinal direction. In fact, plastic yielding circumferentially as well as longitudinally under the test pressure may be considered normal provided no buckling occurs.

Although further analytical and experimental data are needed to *conclusively* establish *this basis for*

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TABLE III

			Stresses at End of Cylinder	
θ	.012θ°	.005θ°	Longitudinal	Circumferential
5°	.060	.025	$\frac{pR}{t}\left(.5+.050\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.029\sqrt{\frac{R}{t}}\right)$
10°	.120	.050	$\frac{pR}{t}\left(.5+.101\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.057\sqrt{\frac{R}{t}}\right)$
15°	.180	.075	$\frac{pR}{t}\left(.5+.153\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.085\sqrt{\frac{R}{t}}\right)$
20°	.240	.100	$\frac{pR}{t}\left(.5+.207\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.113\sqrt{\frac{R}{t}}\right)$
25°	.300	.125	$\frac{pR}{t}\left(.5+.265\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.139\sqrt{\frac{R}{t}}\right)$
30°	.360	.150	$\frac{pR}{t}\left(.5+.326\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.165\sqrt{\frac{R}{t}}\right)$
35°	.420	.175	$\frac{pR}{t}\left(.5+.391\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.189\sqrt{\frac{R}{t}}\right)$
40°	.480	.200	$\frac{pR}{t}\left(.5+.461\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.211\sqrt{\frac{R}{t}}\right)$
45°	.540	.225	$\frac{pR}{t}\left(.5+.536\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.232\sqrt{\frac{R}{t}}\right)$
50°	.600	.250	$\frac{pR}{t}\left(.5+.615\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.251\sqrt{\frac{R}{t}}\right)$
55°	.660	.275	$\frac{pR}{t}\left(5.+.695\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.267\sqrt{\frac{R}{t}}\right)$
60°	.720	.300	$\frac{pR}{t}\left(.5+.770\sqrt{\frac{R}{t}}\right)$	$\frac{pR}{t}\left(1-.278\sqrt{\frac{R}{t}}\right)$
			X	Y

NOTE: This table is correct for *elastic* deformations only.

design, it is supported by available test and experience records.

On this basis the maximum permissible calculated value of $Y\frac{R}{t}$ is 2, i.e., $\left(\frac{2}{Y}\right)^2$ is the largest value of $\left(\frac{R}{t}\right)$ which can be permitted without the installation of a compression ring at the cone cylinder junction. For $\left(\frac{R}{t}\right)$ ratios larger than 2, a compression ring is required with a cross-sectional area of

$$(\tfrac{1}{2}\ tR\tan\theta)\left[\frac{Y\sqrt{\frac{R}{t}}-2}{Y\sqrt{\frac{R}{t}}}\right]$$

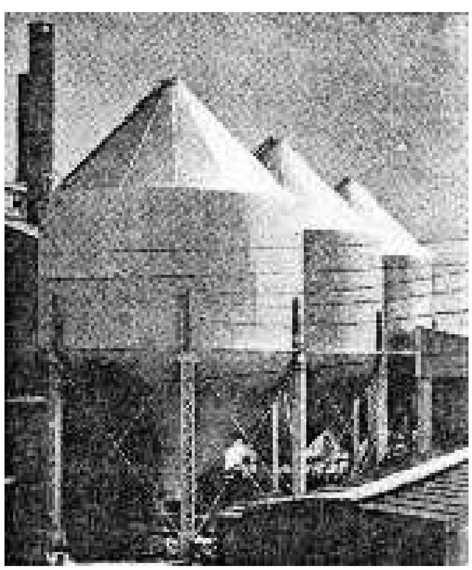
If for"Y" its practical equivalent (.005θ) is used, the expression becomes

$$(\tfrac{1}{2}\ tR\tan\theta)\left[1-\left(\frac{400}{\theta^\circ\sqrt{\frac{R}{t}}}\right)\right]$$

Also, for practical purposes, the maximum (R/t) ratio without a compression ring is $\left(\frac{400}{\theta^\circ}\right)^2$.

If no compression ring is to be used, even though $\sqrt{R/t}$ be larger than 2/Y, the cylinder and cone plates can be made sufficiently thicker than pR/SE and pR/(SEcosθ), respectively (where S=working stress, lbs. Per sq. in., and E-joint efficiency, in decimal form), to keep down to the maximum of " SE" above set, the calculated ring compressive stress at the cone-cylinder junction. In this case the cylinder thickness required is obtained by solving for "T" the cubic equation

$$\frac{pR}{T}\left(1-Y\sqrt{\frac{R}{T}}\right)=-SE$$



Three typical field-erected tanks with cone bottoms and roofs. They are 30 ft. in diam.

The solution may be made easily by trial; or by the safe but approximate equation,

$$T=R\left[-\frac{1}{\frac{1}{3Y}+\sqrt[3]{\frac{SE}{pY}}}\right]^2$$

Of course the corresponding cone thickness $\left(\frac{T}{\cos\theta}\right)$. For "Y" its practical equivalent (.005θ) may be used.

It is recommended that the increased cylinder thickness extend over a shell width (measured longitudinally) of at least $\left(2\tfrac{1}{3}\sqrt{RT}\right)$ inches, and that the increased cone thickness extend over a width (measured along a cone element) of at least $2\tfrac{1}{3}\sqrt{\frac{RT}{\cos\theta}}$ inches.

CAUTION: This analysis is directly applicable only when the cylinder thickness is equal to the cone thickness times (cosθ), and when the pressure is due to gas.

Table IV may be used to plot the curves of Deflection, Rotation, Moment (and corresponding longitudinal stress), Shear and Circumferential Stress for a cylinder strip; the same expressions are *practically* correct for a cone strip provided loads, shears, and deflections be perpendicular to the strip, $\left(\frac{R}{\cos\theta}\right)$ be used instead of (R), and (I) be the actual moment of inertia of the cone strip cross-section. In the Table, (e) is the base of the Natural system of logarithms, and (x) is the distance in inches from the end of the strip.

TABLE IV

1 lb. Load on End of Longitudinal Cylinder Strip

(a) Deflection $=\frac{\cos\beta x}{2e^{\beta x}\,mI\beta^3}$

(b) Rotation $=\frac{1}{2mI\beta^2}\left(\frac{\sin\beta x+\cos\beta x}{e^{\beta x}}\right)$

(c) Moment $=\frac{\sin\beta x}{\beta e^{\beta x}}$

(d) Shear $=\frac{\cos\beta x-\sin\beta x}{e^{\beta x}}$

(e) Circumferential Stress, lbs. per sq. in. =

$$-2\beta\left(\frac{R}{t}\right)\left(\frac{\cos\beta x}{e^{\beta x}}\right)$$

Unit Moment on End of Longitudinal Cylinder Strip

(f) Deflection $=\left(\frac{1}{2mI\beta^2}\right)\left(\frac{\cos\beta x-\sin\beta x}{e^{\beta x}}\right)$

(g) Rotation $=\left(\frac{1}{mI\beta}\right)\left(\frac{\cos\beta x}{e^{\beta x}}\right)$

(h) Moment $=\left(\frac{\sin\beta x+\cos\beta x}{e^{\beta x}}\right)$

(i) Shear $=2\beta\left(\frac{\sin\beta x}{e^{\beta x}}\right)$

(j) Circumferential Stress, lbs. per sq. in. =

$$2\beta^2\left(\frac{R}{t}\right)\left(\frac{\sin\beta x-\cos\beta x}{e^{\beta x}}\right)$$

NOTE: This table is correct for *elastic* deformations only. Angles are in radians.